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The Clebsch–Gordan coefficients of permutation groups S(2)–S(6)

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Abstract. The Clebsch-Gordan (CG) coefficients of the permutation groups S(2)-S(6) are calculated by using the eigenfunction method. Phases have been chosen in a consistent way to exhibit as many symmetries for the coefficients as is possible, including non-multiplicity free cases. The CG coefficients of S(2)-S(5) and those for $[321]\times[321]\rightarrow[321]^5$ of S(6) are tabulated in the square root form of rationals. These tables together with the $S(6)\supset S(5)$ isoscalar factors published earlier provide a complete tabulation of the S(2)-S(6) CG coefficients given in the Young-Yamanouchi basis.

1. Introduction

The Clebsch-Gordan coefficients of the permutation group are useful in many areas. The first application stems from its very definition (Hamermesh 1962), i.e. it is an element of a matrix which transforms the coupled space labelled by two irreducible bases (IRB) of $S(f)$ into a direct sum of irreducible spaces of $S(f)$. The next comes from the fact that the CG coefficients of the permutation group are related to the indirect coupling coefficients for the $SU(mn)\supset SU(m)\times SU(n)$ IRB (Chen *et al* 1978, 1984a). Consequently, the CG coefficients of permutation groups are closely related to the $SU(mn)\supset SU(m)\times SU(n)$ coefficients of fractional parentage (CFP) (Chen 1981, Harvey 1981, Obukhousky *et al* 1982), once the CG coefficients of permutation groups are known, the $SU(mn)\supset SU(m)\times SU(n)$ CFP can easily be calculated. Recently, it has been recognised that the CG coefficients of permutation groups are also related to the indirect coupling coefficients which couple the IRB of the graded unitary groups $U(m/n)$ and $U(p/q)$ into the IRB of $U((mp+nq)/(mq+np))$ (Chen *et al* 1983a).

Vanagas (1972) showed that the ordinary tensor algebra can be extended to the permutation group. The one- and two-body operators can be expressed in terms of the irreducible tensors of the permutation group. The full Racah technique can then be exploited for evaluating the matrix elements of the operators, using the host of the CG coefficients, Racah coefficients, etc of the permutation group. Sullivan (1972) has considered a similar problem involving only the two point partition label of the permutation group.

In recent years, there have been many papers devoted to the CG coefficients of compact groups, for which the reader is referred to a review article by Chen *et al* (1985). The first systematic study of the permutation group CG coefficient is due to Hamermesh (1962). He introduced the K matrix, now known as the $S(f)\supset S(f-1)$ isoscalar factor (ISF) (Chen *et al* 1983c), set up a recursive formula for the K matrix

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and calculated the CG coefficients for the Kronecker product [311]×[311] of the permutation group S(5). Vanagas (1972) studied the CG coefficient for the product $[f-1, 1]\times[f-1, 1]$, the only case which is multiplicity free for a general $S(f)$ and for which an algebraic expression of the $S(f)\supset S(f-1)$ ISF is possible. However, due to the fact that the order of the permutation group $S(f)$ increases drastically with f, the systematic calculation of the permutation group CG coefficients remains a challenging problem. In 1977, two new methods were proposed for computing the CG coefficients. The first one is due to Schindler and Mirman (1977a, b) which is based on the recognition that the columns of the projection matrix are the unnormalised CG vectors, while the second one is due to Chen *et al* (1977) which is based on the fact that the CG vectors of $S(f)$ are the simultaneous eigenvectors of the two-cycle class operators of $S(f)$, $S(f-1)$, ... and S(2). Later, it was shown (Chen and Gao 1982) that the CG vectors of $S(f)$ can be much more easily obtained by diagonalising a single matrix instead of diagonalising simultaneously the $(f-1)$ representation matrices of these $(f-1)$ two-cycle class operators in the Kronecker product space, the former being a suitable linear combination of the latter. Furthermore, with the known CG series for the permutation groups (Itzykson and Nauenberg 1966), the eigenvalues of the matrix can be known beforehand and thus the problem of diagonalising the matrix is reduced to that of solving a set of homogeneous linear equations, which is almost trivial with the help of a computer. Therefore, the formidable problem of computing the CG coefficients of higher-order permutation groups becomes relatively easy. Saharasbudhe *et al* (1981) also proposed a non-genealogical method for calculating the permutation group CG coefficients.

Butler and Wybourne (1976a, b) used a quite different approach to the CG coefficient problem of compact groups. The distinguishing feature of their method is that it requires only a knowledge of character theory and is particularly useful for groups with irreducible representations (irreps) of large dimensions. This method has been applied to the point groups (Butler 1981) and unitary groups (Bickerstaff *et al* 1982), but not yet to the permutation groups.

Two sets of tables of CG coefficients for the permutation groups S(2)–S(6) have been produced. One is computed by a program in Fortran (Schindler and Mirman 1978). Only the CG coefficients for the so-called working triplets of irreps are calculated and listed in the floating point form with 16 decimal places. The bulk of the table is kept in the AIP Document No PAPS JMAPA-18-1697-84, whereas only a very small part, i.e. the CG coefficients for S(2)–S(5), is published in Schindler and Mirman (1977b). The other is computed by a program in Algol-60 based on the eigenfunction method (Chen and Gao 1982). All the CG coefficients of S(2)–S(6), except those for $[321]\times[321]\rightarrow[321]^5$, along with the program are published in a book (Chen and Gao 1981). All the coefficients are listed in square root form of rationals which makes the table much more attractive than the Schindler and Mirman table.

It is well known that the CG coefficients can only be determined up to a unitary transformation in the multiplicity label (Derome 1966, Butler 1975). It was pointed out that the symmetry imposition for the CG coefficient may partially, or in favourable cases even totally remove the ambiguity in the multiplicity separation (Chen and Gao 1981, Chen *et al* 1984b). A serious shortcoming of the Schindler and Mirman result is that the multiplicity separation is entirely arbitrary with the unfavourable consequences that the resulting CG coefficients fail to satisfy certain symmetries, and more seriously, they cannot be written as square roots of simple rationals. If they were able to be put in the form of $(a/b)^{1/2}$, then a and/or b may be extremely large which makes no sense of using the square root form for the coefficients.

The absolute phase of the permutation group CG coefficients tabulated in Hamermesh (1962), Schindler and Mirman (1968) and Chen and Gao (1981) are all chosen randomly. The symmetries of the CG coefficient have been discussed by Hamermesh (1962), Schindler and Mirman (1977a, b), and Butler and Ford (1979). However, the phase factors in the symmetry relations have not been properly considered. This is largely due to the fact that the imposition of the symmetry between the permutation groups and the unitary groups and the homomorphism between U(n) and SU(n) has not been given detailed attention. Sullivan (1983) and Bickerstaff (1984) examined the phase problem in the light of duality between the permutation group and unitary group as well as the simultaneity between U(n) and SU(n).

More recently, a program in Fortran has been written for computing both the $S(f) \supset S(f-1)$ ISF and the $S(f)$ CG coefficient iteratively (Chen *et al* 1984b). A systematic phase convention is adopted for identifying simultaneously the $S(f) \supset S(f-1)$ ISF with the $U(mn) \supset U(m) \times U(n)$ one-particle CFP and the $SU(mn) \supset SU(m) \times SU(n))$ one-particle CFP. Since the $S(f)$ CG coefficient can be expressed in terms of the ISF for $S(f) \supset S(f-1)$, $S(f-1) \supset S(f-2), \ldots$, the absolute phase of the former is totally determined by that of the latter. The phase convention in Chen *et al* (1984b) will be used in this paper.

Summarising, although two sets of tables of CG coefficients for S(2)-S(6) exist, they are both imperfect. The Schindler and Mirman table suffers from the shortcomings that (i) the CG coefficients are lacking in certain symmetries and are in decimal form, (ii) the table is in a too compact form to be used conveniently and (iii) the absolute phase of the CG coefficient is arbitrary. While the Chen and Gao table, although free from the above shortcomings (i) and (ii), is still hampered by the shortcoming (iii). Furthermore, it missed one important case, i.e. the CG coefficients for $[321] \times [321] \rightarrow [321]^5$, which has the highest multiplicity five for the S(6) CG coefficients and is the first interesting case for the permutation group for discussing the non-simple phase representation (Derome 1966, Butler 1975). Barring all these shortcomings, both tables are of rather limited circulation and remain inaccessible for many interested readers. Consequently we feel that it is still worth publishing the CG coefficients table for S(2)-S(6) with consistent phase and as many as possible symmetries and with square root form entries. For the sake of space, in this paper we only publish the CG coefficients for S(2)-S(5) and for $[321] \times [321] \rightarrow [321]^5$ of S(6). By using equation (2.3) given below, the remaining CG coefficients of S(6) are easily obtainable from the S(5) CG coefficients and the $S(6) \supset S(5)$ ISF tabulated in Chen *et al* (1984b).

2. The eigenfunction method

There are two versions of the eigenfunction method for calculating the CG coefficient of permutation groups, one is non-genealogical and the other is genealogical.

2.1. *The non-genealogical method*

According to Chen and Gao (1982), the CG coefficients for the first component, i.e. the one with the maximum Yamanouchi symbol (Hamermesh 1962), of the irrep ν of

S(f), $3 \leqslant f \leqslant 6$, can be found from the following eigenequation (equation (68) in Chen and Gao 1982)

$$\sum_{m_1 m_2} [_{(+)}\langle m_1' m_2' | B | m_1 m_2 \rangle_{(+)} - \lambda \delta_{m_1 m_1'} \delta_{m_2 m_2'}] C^{(\nu)\beta, m=1}_{\sigma m_1, \mu m_2} = 0, \tag{2.1a}$$

$$B = \sum_{n=3}^{f} n C_n, \qquad C_n = \sum_{i \geqslant j = 1}^{n} (ij), \tag{2.1b}$$

where $|m_1 m_2\rangle_{(+)}$ are the basis vectors for the Kronecker product $[\sigma] \times [\mu]$, which belong to the eigenspace of the transposition $C_2 = (12)$ with the eigenvalue equal to $+1$, or equivalently with $D^{[\sigma]}_{m_1 m_1}(12) D^{[\mu]}_{m_2 m_2}(12) = +1$, $C^{[\nu]\beta, m}_{\sigma m_1, \mu m_2}$ are the CG coefficients, $\beta = 1, 2, \ldots, (\sigma\mu\nu)$ is the multiplicity label. From the secular equation of (2.1a), we can determine the eigenvalues λ, which have a one to one correspondence with the maximum Yamanouchi symbols of each irrep, along with their degeneracies which determine the CG series coefficients $(\sigma\mu\nu)$. Conversely, if the CG series of $S(f)$ tabulated by Itzykson and Nauenberg (1966) is used, then both the eigenvalues λ and their degeneracies $(\sigma\mu\nu)$ can be known beforehand without solving the secular equation. With known eigenvalues λ, the linear homogeneous algebraic equation (2.1a) can be easily solved.

From (2.1a) we can get $(\sigma\mu\nu)$ CG vectors for the first component of the irrep ν. The CG vectors for the other components can be calculated successively through the use of the formula

$$C^{(\nu)\beta, \bar{m}}_{\sigma \bar{m}_1, \mu \bar{m}_2} = (D^{(\nu)}_{\bar{m} m}(p_i))^{-1} \sum_{m_1 m_2} (D^{(\sigma)}_{\bar{m}_1 m_1}(p_i) D^{(\mu)}_{\bar{m}_2 m_2}(p_i) - D^{(\nu)}_{mm}(p_i) \delta_{\bar{m}_1 m_1} \delta_{\bar{m}_2 m_2}) C^{(\nu)\beta, m}_{\sigma m_1, \mu m_2} \tag{2.2}$$

where $D^{(\nu)}_{\bar{m} m}(p_i)$ etc, are the Yamanouchi matrix elements of the transposition $p_i = (i-1, i)$ which is properly chosen such that $D^{(\nu)}_{\bar{m} m}(p_i) \neq 0$.

This method is very straightforward. The computer program based on (2.1a) is very short and the calculation can be carried out on a personal computer. The highest order of the linear homogeneous algebraic equation (2.1a) for S(f) with $f \leqslant 6$, is equal to $\frac{1}{2}(16 \times 16) = 128$, corresponding to the Kronecker product $[321] \times [321]$ of S(6).

It should be emphasised that in this method, only the matrices of the $\binom{f}{2}$ transpositions of S(f) are required. The total number of the Yamanouchi matrix elements for the transpositions of S(6) is equal to

$$N_1 = \binom{6}{2} [2(5^2 + 9^2 + 10^2 + 5^2) + 16^2] = 11\,070,$$

where 5, 9, 10, 5 and 16 are the dimensions of the irreps [51], [42], [411], [33], and [321], respectively. In contrast, in the Schindler and Mirman method, all the matrices of the $f!$ permutations of S(f) are required. The total number of the Yamanouchi matrix elements for S(6) is equal to

$$N_2 = (6!)^2 = 518\,400.$$

From the fact that $N_1/N_2 = 2\%$, and considering that the Yamanouchi matrices for the transpositions are the easiest ones to obtain, we conclude that the labour involved in the calculation of the matrix elements in the eigenfunction method is less than 1% of that in the Schindler and Mirman method.

2.2. The genealogical method

The S(f) CG coefficient can be factorised into the S(f) ⊃ S($f-1$) ISF $C^{[\nu]\beta,[\nu']\beta'}_{\sigma\sigma',\mu\mu'}$ and the S($f-1$) CG coefficient $C^{[\nu]\beta',m'}_{\sigma'm_1',\mu'm_2'}$,

$$C^{[\nu]\beta,m}_{\sigma m_1,\mu m_2}=\sum_{\beta'} C^{[\nu]\beta,[\nu']\beta'}_{\sigma\sigma',\mu\mu'} C^{[\nu']\beta',m'}_{\sigma'm_1',\mu'm_2'}. \tag{2.3}$$

The relation between the quantum numbers of S(f) and S($f-1$) are related by the branching rule

$$[\sigma]m_1=[\sigma][\sigma']m_1', \qquad [\mu]m_2=[\mu][\mu']m_2', \qquad [\nu]m=[\nu][\nu']m'. \tag{2.4}$$

The S(f) ⊃ S($f-1$) ISF satisfies an eigenequation (see equation (2.2) in Chen *et al* 1984b).

Equation (2.3) shows that if the S($f-1$) CG coefficient and the S(f) ⊃ S($f-1$) ISF are known, it is trivial to construct the S(f) CG coefficient.

A detailed account of the method for constructing iteratively the S(f) ⊃ S($f-1$) ISF and S(f) CG coefficient is given in Chen *et al* (1984b) and will not be repeated here. The advantage of this method is that the order of the eigenequation satisfied by the S(f) ⊃ S($f-1$) ISF is much lower than that of the eigenequation satisfied by the S(f) CG coefficient. For instance, for the permutation group S(6), the maximum value for the former is only 11, whereas the maximum value for the latter is 128. Therefore the eigenequation for the ISF can be easily solved, and the CG series, CG coefficients and ISF can all be obtained in one stroke. The disadvantage of this method is that the programming is more involved.

3. The phase convention and symmetries

A consistent phase choice is made for the S(f) ⊃ S($f-1$) ISF in Chen *et al* (1984b) to ensure the simultaneity for the U(mn) ⊃ U(m) × U(n) ISF and SU(mn) ⊃ SU(m) × SU(n) ISF (Bickerstaff 1984), which in turn totally fixes the absolute phase of the S(f) CG coefficient to the following effect.

For given irreps σ and μ with the dimensions h_σ and h_μ, respectively, the product basis vectors

$$|m_1m_2\rangle=|\psi^\sigma_{m_1}\psi^\mu_{m_2}\rangle \tag{3.1}$$

are arranged in the ordering of (11), (12), ..., ($1h_\sigma$), (21), (22), ..., ($h_\sigma h_\mu$), whereas the basis vectors $|m_1\rangle$ or $|m_2\rangle$ are arranged in decreasing page order of the corresponding Yamanouchi symbols (Hamermesh 1962). Define a CG vector $x^{[\nu]\beta}$ whose (m_1m_2)th component is

$$(x^{[\nu]\beta})_{m_1m_2}=C^{[\nu]\beta,m}_{\sigma m_1,\mu m_2}. \tag{3.2a}$$

Then the absolute phase of the CG coefficients is determined by demanding the first non-vanishing component of the CG vector $x^{[\nu]\beta}$ be real positive, namely

$$C^{[\nu]\beta,m=1}_{\sigma m_1,\mu m_2}|_{(\underline{m}_1\underline{m}_2)}\rangle 0, \tag{3.2b}$$

where ($\underline{m}_1\underline{m}_2$) means taking m_1 as small as possible followed by taking m_2 as small as possible for which the CG coefficient $C^{[\nu]\beta,1}_{\sigma m_1,\mu m_2}$ is non-zero.

There are two types of symmetries for the permutation group CG coefficient. One is the interchange symmetry and the other is the conjugation (or tilde) symmetry. The interchange symmetry for the $3j$ (or $3jm$) symbols of an arbitrary compact group has been studied by Derome (1966) and Butler (1975). The symmetries of the permutation group CG coefficient have been discussed by Hamermesh (1962), Schindler and Mirman (1977b), Butler and Ford (1979), Chen and Gao (1981).

For simplicity, we use the provisional notation$(ab\,|\,c)_\beta$ for the CG coefficient

$$C^{[\nu_3]\beta,m_3}_{\nu_1m_1,\nu_2m_2}\equiv(ab\,|\,c)_\beta,$$

with $a\equiv\nu_1m$, $b\equiv\nu_2m_2$ and $c\equiv\nu_3m_3$. There are two independent transposition operators π_{12} and π_{23},

$$\pi_{12}(ab\,|\,c)_\beta=(ba\,|\,c)_\beta,\qquad \pi_{23}(ab\,|\,c)_\beta=(ac\,|\,b)_\beta,\tag{3.3a}$$

and two conjugation operators $\mathscr{C}_{12}$ and $\mathscr{C}_{23}$,

$$\mathscr{C}_{12}(ab\,|\,c)_\beta=(\tilde{a}\tilde{b}\,|\,c)_\beta,\qquad \mathscr{C}_{23}(ab\,|\,c)_\beta=(a\tilde{b}\,|\,\tilde{c})_\beta.\tag{3.3b}$$

Notice that here 1, 2 and 3 always refer to the positions numbered as in $C^{\nu_3m_3}_{\nu_1m_1,\nu_2m_2}$, rather than to the indices a, b and c, respectively. Hence $\pi_{12}(bc\,|\,a)=(cb\,|\,a)$, and $\pi_{23}(cb\,|\,a)=(ca\,|\,b)$. The transpositions π_{12} and π_{23} generate the permutation group $\mathscr{S}(3)$, while the conjugation operators $\mathscr{C}_{12}$ and $\mathscr{C}_{23}$ generate the four-group $V=(e,\mathscr{C}_{12},\mathscr{C}_{23},\mathscr{C}_{13})$. It is easily seen that

$$[\pi_{ij},\mathscr{C}_{ij}]=0,\qquad \text{for } ij=12, 23, 13,\tag{3.3c}$$

whereas

$$[\pi_{12},\mathscr{C}_{23}]\neq0,\qquad [\pi_{12},\mathscr{C}_{13}]\neq0.\tag{3.3d}$$

(3.3c) and (3.3d) show that the interchange and tilde symmetries in general do not commute, but those referring to the same pair of indices do commute.

Corresponding to the symmetries of the $S(f)\supset S(f-1)$ ISF given by (4.4), (4.5) and (4.6) in Chen *et al* (1984b), the CG coefficient of $S(f)$ has the following symmetries.

(1)

$$C^{[\nu]\beta,m}_{\sigma m_1,\mu m_2}=\varphi_4(\sigma\mu\nu_\beta)\Lambda^{\sigma}_{m_1}\Lambda^{\mu}_{m_2}C^{[\nu]\beta,m}_{\tilde{\sigma}\tilde{m}_1,\tilde{\mu}\tilde{m}_2}\tag{3.4a}$$

$$=\varphi_5(\sigma\mu\nu_\beta)\Lambda^{\sigma}_{m_1}\Lambda^{\nu}_{m}C^{[\tilde{\nu}]\beta,\tilde{m}}_{\tilde{\sigma}\tilde{m}_1,\mu m_2}\tag{3.4b}$$

$$=\varphi_6(\sigma\mu\nu_\beta)\Lambda^{\mu}_{m_2}\Lambda^{\nu}_{m}C^{[\tilde{\nu}]\beta,\tilde{m}}_{\sigma m_1,\tilde{\mu}\tilde{m}_2},\tag{3.4c}$$

where Λ^{ν}_{m} etc, are the sign factors defined by Hamermesh (1962). The phase factors $\varphi_i(\sigma\mu\nu_\beta)$ are dictated by the phase convention (3.2b). They can be expressed as

$$\varphi_4(\sigma\mu\nu_\beta)=\operatorname{sign}(\Lambda^{\sigma}_{m_1}\Lambda^{\mu}_{m_2}C^{[\nu]\beta,1}_{\sigma m_1,\mu m_2}|(\bar{m}_1\bar{m}_2)),\tag{3.5a}$$

$$\varphi_5(\sigma\mu\nu_\beta)=\operatorname{sign}(\Lambda^{\sigma}_{m_1}\Lambda^{\nu}_{m}C^{[\nu]\beta,h\nu}_{\sigma m_1,\mu m_2}|(\bar{m}_1\underline{m}_2)),\tag{3.5b}$$

$$\varphi_6(\sigma\mu\nu_\beta)=\operatorname{sign}(\Lambda^{\mu}_{m_2}\Lambda^{\nu}_{m}C^{[\nu]\beta,h\nu}_{\sigma m_1,\mu m_2}|(\underline{m}_1\bar{m}_2)),\tag{3.5c}$$

where $(\bar{m}_1\bar{m}_2)$ means first taking m_1 as large as possible and then taking m_2 as large as possible, whereas $(\bar{m}_1\underline{m}_2)$ means first taking m_1 as large as possible and then taking m_2 as small as possible. The meaning of $(\underline{m}_1\bar{m}_2)$ is opposite to that of $(\bar{m}_1\underline{m}_2)$.

Notice that for non-multiplicity free cases, (3.4a) ((3.4b) or (3.4c)) in general only holds when the partitions $[\sigma]$ and $[\mu]$ ($[\sigma]$ and $[\nu]$, or $[\mu]$ and $[\nu]$) are not self-conjugate simultaneously. If both are self-conjugate (for S(f) with $f \leqslant 7$, then they must be equal), the imposition of the symmetry

$$C^{[\nu]\beta,m}_{\sigma m_1,\sigma m_2} = \pm \Lambda^{\sigma}_{m_1} \Lambda^{\sigma}_{m_2} C^{[\nu]\beta,m}_{\sigma \tilde{m}_1,\sigma \tilde{m}_2}, \qquad \text{for } [\sigma]=[\tilde{\sigma}] \neq [\nu], \tag{3.4d}$$

or

$$C^{[\nu]\beta,m}_{\sigma m_1,\nu m_2} = \pm \Lambda^{\nu}_{m_2} \Lambda^{\nu}_{m} C^{[\nu]\beta,\tilde{m}}_{\sigma m_1,\nu \tilde{m}_2}, \qquad \text{for } [\nu]=[\tilde{\nu}] \neq [\sigma], \tag{3.4d'}$$

will be of help for the multiplicity separation.

The sign factor Λ^{ν}_{m} can be factorised as (Butler and Ford 1979)

$$\Lambda^{\nu}_{m} = \Lambda^{\nu}_{\nu'} \Lambda^{\nu'}_{m'}, \qquad \Lambda^{\nu}_{\nu'} = (-1)^{n_f}, \tag{3.6}$$

where n_f is the number of squares below the square labelled with f in the Young tableau Y^{ν}_{m} of S(f). The phase factors ε_4–ε_6 for the symmetry relation of S(f) ⊃ S($f-1$) (see equation (4.4) in Chen *et al* 1984b) can be expressed as

$$\begin{aligned} \varepsilon_4 &= \Lambda^{\sigma}_{\sigma'} \Lambda^{\mu}_{\mu'} \varphi_4(\sigma\mu\nu_\beta)\varphi_4(\sigma'\mu'\nu'_{\beta'}), \\ \varepsilon_5 &= \Lambda^{\sigma}_{\sigma'} \Lambda^{\nu}_{\nu'} \varphi_5(\sigma\mu\nu_\beta)\varphi_5(\sigma'\mu'\nu'_{\beta'}), \\ \varepsilon_6 &= \Lambda^{\mu}_{\mu'} \Lambda^{\nu}_{\nu'} \varphi_6(\sigma\mu\nu_\beta)\varphi_6(\sigma'\mu'\nu'_{\beta'}). \end{aligned} \tag{3.7}$$

It must be pointed out that the relation (4.7) in Chen *et al* (1984b) for ε_4–ε_6 should be corrected by the above equation (3.7).

(2) For $\sigma \neq \mu$,

$$C^{[\nu]\beta,m}_{\sigma m_1,\mu m_2} = \varphi_7(\sigma\mu\nu_\beta) C^{[\nu]\beta,m}_{\mu m_2,\sigma m_1}, \tag{3.8a}$$

$$\varphi_7(\sigma\mu\nu_\beta) = \text{sign}(C^{[\nu]\beta,1}_{\sigma m_1,\mu m_2} | (\underline{m}_2 \underline{m}_1)), \tag{3.8b}$$

where $(\underline{m}_2 \underline{m}_1)$ means that we first take m_2 as small as possible and then take m_1 as small as possible.

For $\sigma = \mu$,

$$C^{[\nu]\beta,m}_{\sigma m_1,\sigma m_2} = \delta_{\nu_\beta} C^{[\nu]\beta,m}_{\sigma m_2,\sigma m_1}. \tag{3.8c}$$

The irrep $[\nu]\beta$ is said to belong to the symmetric product $[\sigma \times \mu]_s$ for $\delta_{\nu_\beta} = +1$, or antisymmetric product $[\sigma \times \mu]_a$ for $\delta_{\nu_\beta} = -1$.

(3) For $\sigma \neq \mu \neq \nu \neq \sigma$

$$\begin{aligned} \left(\frac{1}{h_\nu}\right)^{1/2} C^{[\nu]\beta,m}_{\sigma m_1,\mu m_2} &= \varphi_8(\sigma\mu\nu_\beta)\left(\frac{1}{h_\sigma}\right)^{1/2} C^{[\sigma]\beta,m_1}_{\nu m,\mu m_2} \\ &= \varphi_9(\sigma\mu\nu_\beta)\left(\frac{1}{h_\mu}\right)^{1/2} C^{[\mu]\beta,m_2}_{\sigma m_1,\nu m}, \end{aligned} \tag{3.9a}$$

$$\varphi_8(\sigma\mu\nu_\beta) = \text{sign}(C^{[\nu]\beta,m}_{\sigma 1,\mu m_2} | (\underline{m}\underline{m}_2)), \tag{3.9b}$$

$$\varphi_9(\sigma\mu\nu_\beta) = \text{sign}(C^{[\nu]\beta,m}_{\sigma m_1,\mu 1} | (\underline{m}_1 \underline{m})). \tag{3.9c}$$

For $\sigma = \mu \neq \nu$, (3.9a) still holds. Besides we also have,

$$C^{[\sigma]\beta,m_2}_{\sigma m_1,\nu m} = \delta_{\nu_\beta} C^{[\sigma]\beta,m_1}_{\sigma m_2,\nu m} \tag{3.9d}$$

which can be easily proved by using (3.8c) and (3.9a).

(4) For $\sigma = \mu = \nu$

The discussion of the $3j$ symbol symmetries by Derome (1966) can be easily carried over to the CG coefficient. Let

$$(ab|c)_\beta = C^{[\sigma]\beta,m_3}_{\sigma m_1,\sigma m_2},\ \beta = 1, 2, \ldots, (\sigma\sigma\sigma)$$

be regarded as functions of a, b and c, which carry a $(\sigma\sigma\sigma)$-dimensional representation $\mathscr{D}$ for the permutation group $\mathscr{S}(3)$ defined by (3.3a). According to Derome (1966), if $\mathscr{D}$ does not contain the irrep [21], then σ is said to be a simple phase representation, otherwise, a non-simple phase representation. For the latter case, it is impossible to choose all the CG coefficients such that the effect of a permutation of m_i's is at most a multiplicative phase.

The irrep [321] of S(6) is a non-simple phase representation. The five-dimensional representation $\mathscr{D}$ of $\mathscr{S}(3)$ associated with it contains the irreps [3], [21] and $[1^3]$ twice, once and once, respectively (Butler 1975). Hence for [321], we may choose

$$\pi(ab|c)_\beta = (ab|c)_\beta, \qquad \beta = 1, 2, \tag{3.10a}$$

$$\pi(ab|c)_\beta = \delta_\pi(ab|c)_\beta, \qquad \beta = 3, \tag{3.10b}$$

$$\pi(ab|c)_\beta = \sum_{\alpha=4}^{5} D^{[21]}_{\alpha\beta}(\pi)(ab|c)_\alpha, \qquad \beta = 4, 5, \tag{3.10c}$$

where δ_π is the parity of the permutation π.

Due to (3.3d), it is impossible to give simultaneously a simple structure (i.e. a diagonal form) to both the matrices describing the interchange and conjugate symmetries. Therefore, instead of imposing the symmetries (3.10) for the irrep [321], we require that

$$\pi_{12}(ab|c)_\beta = \delta_\beta(ab|c)_\beta, \qquad \mathscr{C}_{12}(ab|c)_\beta = \Delta_\beta\Lambda_a\Lambda_b(ab|c)_\beta, \tag{3.11}$$

for $\beta = 1, 2, \ldots, 5$, where $\delta_\beta = \pm 1$ and $\Delta_\beta = \pm 1$ (see (4.8)).

For multiplicity free cases the symmetries (3.4)-(3.11) are satisfied automatically, however the phases now enter of necessity instead of being determined by phase convention. For non-multiplicity free cases, the symmetries (3.4d), (3.8c), (3.9d) and (3.11) are imposed to reduce the arbitrariness in the multiplicity separation and to get simpler numerical values for the CG coefficients which is crucial for the tabulation in square root form of rationals, while the other symmetry relations in (3.4)-(3.9) are used to find the CG coefficients which are not tabulated.

Now we use $[41]\times[32]\rightarrow[221]$ as an example for determining the phase factors φ_4-φ_9 from the CG coefficient table 3.2 and the Λ^ν_m table 5 in § 4.

$$\varphi_4([41][32][221]) = \mathrm{sign}(\Lambda^{[41]}_4\Lambda^{[32]}_3 C^{[221]1}_{[41]4,[32]3}) = -1,$$

$$\varphi_5([41][32][221]) = \mathrm{sign}(\Lambda^{[41]}_4\Lambda^{[221]}_5 C^{[221]5}_{[41]4,[32]2}) = -1,$$

$$\varphi_6([41][32][221]) = \mathrm{sign}(\Lambda^{[32]}_5\Lambda^{[221]}_5 C^{[221]5}_{[41]3,[32]5}) = -1,$$

$$\varphi_7([41][32][221]) = \mathrm{sign}(C^{[221]1}_{[41]3,[32]1}) = -1,$$

$$\varphi_8([41][32][221]) = \mathrm{sign}(C^{[221]1}_{[41]1,[32]4}) = +1,$$

$$\varphi_g([41][32][221]) = \mathrm{sign}(C^{[221]1}_{[41]3,[32]1}) = -1.$$

4. Tables of the CG coefficients

The CG coefficients of S(3)-S(6) have been calculated by both the non-genealogical and genealogical versions of the eigenfunction method and checked one against another.

The CG coefficients of S(3)-S(5) and those for $[321]\times[321]\rightarrow[321]^5$ of S(6) are listed in tables 1-4, while the sign factors Λ^{ν}_{m} are listed in table 5.

(1) The partitions are arranged in the order of decreasing row symmetry from top to bottom in the corresponding Young diagrams. We only listed the CG coefficients for the products $[\sigma]\times[\mu]$ where $[\mu]$ is below $[\sigma]$ and $[\sigma]$ is no lower than the self-conjugate partition. The other CG coefficients can be found by using the symmetries (3.4)-(3.9).

(2) Each table (except table 4) for given σ and μ is divided into two subtables according to the eigenvalues $\varkappa$ of the transposition (12), the upper one corresponding to $\varkappa=+1$, while the lower one corresponding to $\varkappa=-1$.

(3) In tables 1-3.6, the second column gives the normalisation factor for each row vector. All the entries in tables 1-4 are the square values of the CG coefficients, with an asterisk denoting a negative coefficient and blank a null coefficient.

(4) For the tables with $\sigma=\mu$, the subscript s(a) attached to the partition $[\nu]\beta$ in the table heading indicates that the irrep $[\nu]\beta$ belongs to the symmetric (antisymmetric) product. The multiplicity label is denoted by $\alpha, \beta, \ldots, \varepsilon$.

(5) Tables 4.1-4.4 only give a quarter of the whole table for $[321]\times[321]\rightarrow[321]^5$. The remaining coefficients can be found from the relation (4.8) given below.

(6) The CG coefficients of S(6) other than those for $[321]\times[321]\rightarrow[321]^5$ can be found from the S(5) CG coefficients and $S(6)\supset S(5)$ ISF tabulated in Chen *et al* (1984b) by using (2.3).

(7) The following symmetries are to be noted for the CG coefficients under some special cases.

(a)
$$C^{[22]c}_{[22]a,[22]b} = -\Lambda^{[22]}_{a}\Lambda^{[22]}_{c}C^{[22]\bar{c}}_{[22]\bar{a},[22]b}$$
$$= -\Lambda^{[21]}_{a}\Lambda^{[21]}_{b}C^{[22]c}_{[22]\bar{a},[22]\bar{b}} \qquad (4.1)$$

(b)
$$C^{[311]c}_{[311]a,[41]b} = -\Lambda^{[311]}_{a}\Lambda^{[311]}_{c}C^{[311]\bar{c}}_{[311]\bar{a},[41]b} \qquad (4.2)$$

(c)
$$C^{[311]\tau,c}_{[311]a,[32]b} = \pm\Lambda^{[311]}_{a}\Lambda^{[311]}_{c}C^{[311]\tau,\bar{c}}_{[311]\bar{a},[32]b}, \qquad \text{for } \tau=\begin{cases}\alpha\\ \beta\end{cases} \qquad (4.3)$$

(d)
$$C^{[321]\tau,c}_{[51]a,[321]b} = C^{[321]\tau,b}_{[51]a,[321]c}, \qquad \text{for } \tau=\alpha,\beta$$
$$= \mp\Lambda^{[321]}_{b}\Lambda^{[321]}_{c}C^{[321]\tau,\bar{c}}_{[51]a,[321]\bar{b}}, \qquad \text{for } \tau=\begin{cases}\alpha\\ \beta\end{cases} \qquad (4.4)$$

(e)
$$C^{[321]\tau,c}_{[33]a,[321]b} = \pm C^{[321]\tau,b}_{[33]a,[321]c}, \qquad \text{for } \tau=\begin{cases}\alpha\\ \beta\end{cases}$$
$$= \pm\Lambda^{[321]}_{b}\Lambda^{[321}_{c}C^{[321]\tau,\bar{c}}_{[33]a,[321]\bar{b}}, \qquad \text{for } \tau=\begin{cases}\alpha\\ \beta\end{cases} \qquad (4.5)$$

(f)
$$C^{[321]\tau,c}_{[42]a,[321]b} = C^{[321]\tau,b}_{[42]a,[321]c}, \qquad \text{for } \tau=\alpha,\beta,\gamma$$
$$= \mp\Lambda^{[321]}_{b}\Lambda^{[321]}_{c}C^{[321]\tau,\bar{c}}_{[42]a,[321]\bar{b}}, \qquad \text{for } \tau=\begin{cases}\alpha,\gamma\\ \beta\end{cases} \qquad (4.6)$$

(g)
$$C^{[321]\tau,c}_{[411]a,[321]b} = \pm C^{[321]\tau,b}_{[411]a,[321]c}, \qquad \text{for } \tau=\begin{cases}\alpha\\ \beta,\gamma,\delta\end{cases}$$
$$= \mp\Lambda^{[321]}_{b}\Lambda^{[321]}_{c}C^{[321]\tau,\bar{c}}_{[411]a,[321]\bar{b}}, \qquad \text{for } \tau=\begin{cases}\alpha,\gamma\\ \beta,\delta\end{cases} \qquad (4.7)$$

(h) $$C^{[321]\tau,c}_{[321]a,[321]b} = \pm C^{[321]\tau,c}_{[321]b,[321]a}, \qquad \text{for } \tau = \begin{cases} \alpha, \delta, \varepsilon \\ \beta, \gamma \end{cases}$$

$$= \mp \Lambda^{[321]}_{a} \Lambda^{[321]}_{b} C^{[321]\tau,c}_{[321]\bar{a},[321]\bar{b}}, \qquad \text{for } \tau = \begin{cases} \alpha, \gamma \\ \beta, \delta, \varepsilon \end{cases} \tag{4.8}$$

(8) The meaning of the table heading is as follows

	N \	$(m_1 m_2)$
$[\nu]m$		

Table 1. The CG coefficients of permutation groups.

1. [21]×[21]=[3]s+[21]s+[1³]a.

	N \	(11)	(22)
[3]	2	1	1
[21]1	2	1	*1
	N \	(12)	(21)
[21]2	2	*1	*1
[1³]	2	1	*1

2.1. [31]×[31]=[4]s+[31]s+[22]s+[211]a.

	N \	(11)	(12)	(21)	(22)	(33)
[4]	3	1			1	1
[31]1	6	4			*1	*1
2	6		*1	*1	2	*2
[22]1	6		2	2	1	*1
[211]1	2		1	*1		
	N \	(13)	(23)	(31)	(32)	
[31]3	6	*1	*2	*1	*2	
[22]2	6	2	*1	2	*1	
[211]2	2	1		*1		
3	2		1		*1	

2.2. [31]×[22]=[31]+[211].

	N \	(11)	(21)	(32)
[31]1	2		1	1
2	4	2	1	*1
[211]1	4	2	*1	1
	N \	(12)	(22)	(31)
[31]3	4	2	*1	*1
[211]2	4	2	1	1
3	2		*1	1

2.3. [22]×[22]=[4]s+[22]s+[1⁴]a.

	N \	(11)	(22)
[4]	2	1	1
[22]1	2	1	*1
	N \	(12)	(21)
[22]2	2	*1	*1
[1⁴]	2	1	*1

3.1. [41]×[41]=[5]s+[41]s+[32]s+[311]a.

	N	(11)	(12)	(13)	(21)	(22)	(23)	(31)	(32)	(33)	(44)
[5]	4	1				1				1	1
[41]1	12	9				*1				*1	*1
2	36		*3		*3	20				*5	*5
3	36			*3			*5	*3	*5	10	*10
[32]1	36		15		15	4				*1	*1
2	36			15			*1	15	*1	2	*2
4	6						2		2	1	*1
[311]1	2		1		*1						
2	2			1				*1			
4	2						1		*1		

	N	(14)	(24)	(34)	(41)	(42)	(43)
[41]4	36	*3	*5	*10	*3	*5	*10
[32]3	36	15	*1	*2	15	*1	*2
5	6		2	*1		2	*1
[311]3	2	1			*1		
5	2		1			*1	
6	2			1			*1

3.2a. [41]×[32]=[41]+[32]+[311]+[221].

	N	(11)	(12)	(14)	(21)	(22)	(24)	(31)	(32)	(34)	(43)	(45)
[41]1	3				1				1		1	
2	45	15			4				*1	12	*1	12
3	45		15			*1	12	*1	2	6	*2	*6
[32]1	72	12			20				*5	*15	*5	*15
2	144		24			*10	*30	*10	20	*15	*20	15
4	48			*18		*10		*10	*5		5	
[311]1	40	20			*12				3	*1	3	*1
2	80		40			6	*2	6	*12	*1	12	1
4	16					*2	6	2		*3		3
[221]1	16			10		*2		*2	*1		1	
3	16					6	2	*6		*1		1

3.2b. [41]×[32]=[41]+[32]+[311]+[221].

	N	(13)	(15)	(23)	(25)	(33)	(35)	(41)	(42)	(44)
[41]4	45	15		*1	12	*2	*6	*1	*2	*6
[32]3	144	24		*10	*30	*20	15	*10	*20	15
5	48		*18	*10		5		*10	5	
[311]3	80	40		6	*2	12	1	6	12	1
5	16			*2	6		3	2		3
6	8					*1	*3		1	3
[221]2	16		10	*2		1		*2	1	
4	16			6	2		1	*6		1
5	8					3	*1		*3	1

3.3a. [41]×[311]=[41]+[32]+[311]+[221]+[21^3].

	N	(11)	(12)	(14)	(21)	(22)	(24)	(31)	(32)	(34)	(43)	(45)	(46)
[41]1	3				1				1		1		
2	3	*1								1		1	
3	3		*1				*1						1
[32]1	48	20			*12				3	5	3	5	
2	48		20			3	*5	3	*6		6		5
4	96					*2	30	*2	*1	*15	1	15	30
[311]1	48	12			20				*5	3	*5	3	
2	48		12			*5	*3	*3	10		*10		3
4	48			*12		*3	5	3		10		*10	5
[221]1	96					30	2	30	15	*1	*15	1	2
3	48			20		5	3	*5		6		*6	3
[21^3]1	3			1		*1		1					

3.3b. [41]×[311]=[41]+[32]+[311]+[221]+[21^3].

	N	(13)	(15)	(16)	(23)	(25)	(26)	(33)	(35)	(36)	(41)	(42)	(44)
[41]4	3	*1				*1				*1			
[32]3	48	20			3	*5		6		*5	3	6	
5	96				*2	30		1	15	*30	*2	1	15
[311]3	48	12			*5	*3		*10		*3	*5	*10	
5	48		*12		*3	5			*10	*5	3		*10
6	48			*12			*20	*3	*5			3	5
[221]2	96				30	2		*15	1	*2	30	*15	1
4	48		20		5	3			*6	*3	*5		*6
5	48			20			*12	5	*3			*5	3
[21^3]2	3		1		*1						1		
3	3			1				*1				1	
4	3						1		*1				1

3.4a. [32] × [32] = [5]s + [41]s + [32]s + [311]a + [221]s + [21^3]a.

	N	(11)	(12)	(14)	(21)	(22)	(24)	(33)	(35)	(41)	(42)	(44)	(53)	(55)
[5]	5	1				1		1				1		1
[41]1	30	4				4		4				*9		*9
2	18	4				*1	*3	*1	*3		*3		*3	
3	36		*2	*6	*2	4	*3	*4	3	*6	*3		3	
[32]1	36	16				*4	3	*4	3		3		3	
2	72		*8	6	*8	16	3	*16	*3	6	3		*3	
4	24		2		2	1		*1				9		*9
[311]1	4						1		1		*1		*1	
2	8			2			1		*1	*2	*1		1	
4	40		8	6	*8		*3		3	*6	3		*3	
[221]1	8		2		2	1		*1				*1		1
3	8			2			*1		1	2	*1		1	
[21^3]1	20		6	*2	*6		1		*1	2	*1		1	

3.4b. [32] × [32] = [5]s + [41]s + [32]s + [311]a + [221]s + [21^3]a.

	N	(13)	(15)	(23)	(25)	(31)	(32)	(34)	(43)	(45)	(51)	(52)	(54)
[41]4	36	*2	*6	*4	3	*2	*4	3	3		*6	3	
[32]3	72	*8	6	*16	*3	*8	*16	*3	*3		6	*3	
5	24	2		*1		2	*1			*9			*9
[311]3	8		2		*1			*1	1		*2	1	
5	40	8	6		3	*8		3	*3		*6	*3	
6	20			4	*3		*4	3	*3			3	
[221]2	8	2		*1		2	*1			1			1
4	8		2		1			1	1		2	1	
5	4				*1			1	1			*1	
[21^3]2	20	6	*2		*1	*6		*1	1		2	1	
3	10			3	1		*3	*1	1			*1	
4	2									1			*1

3.5a. $[32]\times[311]=[41]+[32]+[311]^2+[221]+[21^3]$.

		N	(11)	(12)	(14)	(21)	(22)	(24)	(33)	(35)	(36)	(41)
[41]	1	3	1				1		1			
	2	60	*12				3	*5	3	*5		
	3	120		6	10	6	*12		12		*10	*2
[32]	1	24						4		4		
	2	48			*8						8	10
	4	48		*10	*6	*10	*5	3	5	3	*6	
[311α]	1	40	16				*4		*4			
	2	80		*8		*8	16		*16			*6
	4	80			*8			*16		16	*8	10
[311β)	1	60	4				*1	*15	*1	*15		
	2	60		*1	15	*1	2		*2		*15	12
	4	60		15	1	*15		2		*2	1	
[221]	1	48		6	*10	6	3	5	*3	*5	*10	
	3	48		*8		8						*6
$[21^3]$	1	120		10	*6	*10		*12		12	*6	*30

		N	(42)	(44)	(53)	(55)	(56)
[41]	1	3					
	2	60	*1	15	*1	15	
	3	120	*1	*15	1	15	*30
[32]	1	24	5	3	5	3	
	2	48	5	*3	*5	3	*6
	4	48					
[311α)	1	40	*3	5	*3	5	
	2	80	*3	*5	3	5	*10
	4	80	*5	3	5	*3	*6
[311β]	1	60	12		12		
	2	60	6		*6		
	4	60		6		*6	*12
[221]	1	48					
	3	48	3	5	*3	*5	*10
$[21^3]$	1	120	15	*1	*15	1	2

3.5b. [32]×[311]=[41]+[32]+$[311]^2$+[221]+$[31^3]$.

		N	(13)	(15)	(16)	(23)	(25)	(26)	(31)	(32)	(34)	(43)
[41]	4	120	6	10		12		10	6	12		1
[32]	3	48		*8				*8				*5
	5	48	*10	*6		5	*3	6	*10	5	*3	
[311α]	3	80	*8			*16			*8	*16		3
	5	80		*8			16	8			16	5
	6	40			16		4				*4	5
[311β]	3	60	*1	15		*2		15	*1	*2		*6
	5	60	15	1			*2	*1	*15		*2	
	6	60			*4	15	*1			*15	1	
[221]	2	48	6	*10		*3	*5	10	6	*3	*5	
	4	48	*8						8			*3
	5	24				*4				4		*3
$[21^3]$	2	120	10	*6			12	6	*10		12	*15
	3	60			12	5	3			*5	*3	*15
	4	3			1		*1				1	

		N	(45)	(46)	(51)	(52)	(54)
[41]	4	120	15	30	*2	1	15
[32]	3	48	3	6	10	*5	3
	5	48					
[311α]	3	80	5	10	*6	3	5
	5	80	*3	6	10	5	*3
	6	40	3			*5	*3
[311β]	3	60			12	*6	
	5	60	*6	12			*6
	6	60	12				*12
[221]	2	48					
	4	48	*5	10	*6	*3	*5
	5	24	5			3	*5
$[21^3]$	2	120	1	*2	*30	*15	1
	3	60	*1			15	1
	4	3					

3.6a. $[311]\times[311]=[5]_s+[41]_s+[32]_s^2+[311]_a+[221\alpha]_s+[221\beta]_a+[21^3]_a+[1^5]_s$.

		N	(11)	(12)	(14)	(21)	(22)	(24)	(33)	(35)	(36)	(41)
[5]		6	1				1		1			
[41]	1	6	1				1		1			
	2	72	20				*5	*3	*5	*3		
	3	72		*5	3	*5	10		*10		*3	3
[32α]	1	12	4				*1		*1			
	2	12		*1		*1	2		*2			
	4	96		*6	10	*6	*3	*5	3	5	10	10
[32β]	1	72	4				*1	15	*1	15		
	2	72		*1	*15	*1	2		*2		15	*15
	4	12		2		2	1		*1			
[311]	1	4						1		1		
	2	4			*1						1	1
	4	4		1		*1						
[221α]	1	96		10	6	10	5	*3	*5	3	6	6
	3	12			*1			*2		2	*1	*1
[221β]	1	12			2			*1		1	2	*2
	3	72		*15	1	15		2		*2	1	*1
[21³]	1	72		3	5	*3		10		*10	5	*5

		N	(42)	(44)	(53)	(55)	(56)	(63)	(65)	(66)
[5]		6		1		1				1
[41]	1	6		*1		*1				*1
	2	72	*3	5	*3	5				*20
	3	72		10		*10	*5	*3	*5	
[32α]	1	12		*1		*1				4
	2	12		*2		2	1		1	
	4	96	*5	3	5	*3	6	10	6	
[32β]	1	72	15	1	15	1				*4
	2	72		2		*2	*1	15	*1	
	4	12		1		*1	2		2	
[311]	1	4	*1		*1					
	2	4						*1		
	4	4					1		*1	
[221α]	1	96	*3	*5	3	5	*10	6	*10	
	3	12	*2		2			*1		
[221β]	1	12	1		*1			*2		
	3	72	*2		2		15	*1	*15	
[21³]	1	72	*10		10		*3	*5	3	

3.6b. $[311]\times[311]=[5]_s+[41]_s+[32]_s^2+[221\alpha]_s+[221\beta]_a+[21^3]_a+[1^5]_s$.

	N	(13)	(15)	(16)	(23)	(25)	(26)	(31)	(32)	(34)	(43)
[41] 4	72	*5	3		*10		3	*5	*10		
[32α] 3	12	*1			*2			*1	*2		
5	96	*6	10		3	5	*10	*6	3	5	5
[32β] 3	72	*1	*15		*2		*15	*1	*2		
5	12	2			*1			2	*1		
[311] 3	4		*1				*1				
5	4	1						*1			
6	4				1				*1		
[221α] 2	96	10	6		*5	3	*6	10	*5	3	3
4	12		*1			2	1			2	2
5	12			4		1				*1	*1
[221β] 2	12		2			1	*2			1	*1
4	72	*15	1			*2	*1	15		*2	2
5	72			*4	*15	*1			15	1	*1
[21³] 2	72	3	5			*10	*5	*3		*10	10
3	72			*20	3	*5			*3	5	*5
4	6			1		*1				1	*1
[1⁵]	6			1		*1				1	1

	N	(45)	(46)	(51)	(52)	(54)	(61)	(62)	(64)
[41] 4	72	*10	5	3		*10		3	5
[32α] 3	12	2	*1			2			*1
5	96	*3	*6	10	5	*3		*10	*6
[32β] 3	72	*2	1	*15		*2		*15	1
5	12	*1	*2			*1			*2
[311] 3	4			1				1	
5	4		*1						1
6	4	1				*1			
[221α] 2	96	5	10	6	3	5		*6	10
4	12			*1	2			1	
5	12				1		4		
[221β] 2	12			*2	*1			2	
4	72		*15	*1	2			1	15
5	72	15			1	*15	4		
[21³] 2	72		3	*5	10			5	*3
3	72	*3			5	3	20		
4	6				1		*1		
[1⁵]	6				*1		1		

4.1. $[321]\times[321]\rightarrow[321\alpha]_s+[321\beta]_a+[321\gamma]_a+[321\delta]_s+[321\varepsilon]_s$. (to be continued)

$[\nu]\beta$	$[321]\alpha$								$[321]\beta$								$[321]\gamma$			
m_1, m_2 \ m	1	2	4	6	7	9	12	14	1	2	4	6	7	9	12	14	1	2	4	6
1, 1	$\frac{1}{30}$																			
2		*$\frac{1}{120}$	$\frac{1}{160}$				*$\frac{3}{32}$							*$\frac{1}{24}$						
4		$\frac{1}{160}$						*$\frac{3}{32}$					*$\frac{5}{96}$	*$\frac{1}{32}$						
6				*$\frac{1}{30}$																
7			$\frac{5}{384}$		$\frac{1}{120}$	*$\frac{1}{32}$	*$\frac{1}{128}$	$\frac{1}{96}$			$\frac{5}{96}$				*$\frac{1}{32}$	$\frac{1}{24}$			$\frac{5}{384}$	
9		$\frac{1}{96}$	$\frac{1}{128}$		*$\frac{1}{32}$		$\frac{5}{384}$			$\frac{1}{24}$	$\frac{1}{32}$				$\frac{5}{96}$			$\frac{1}{96}$	$\frac{1}{128}$	
12					*$\frac{1}{32}$	$\frac{5}{96}$							$\frac{1}{32}$	*$\frac{5}{96}$				*$\frac{3}{32}$		
14					$\frac{1}{24}$								*$\frac{1}{24}$						*$\frac{3}{32}$	
2, 1		*$\frac{1}{120}$	$\frac{1}{160}$				*$\frac{3}{32}$							$\frac{1}{24}$						
2	*$\frac{1}{120}$	$\frac{1}{60}$	$\frac{1}{320}$				*$\frac{3}{64}$													
4	$\frac{1}{160}$	$\frac{1}{320}$						$\frac{3}{64}$				*$\frac{5}{96}$	*$\frac{5}{192}$	$\frac{1}{64}$						$\frac{5}{96}$
6			$\frac{5}{384}$		$\frac{1}{120}$	$\frac{1}{32}$	*$\frac{1}{128}$	*$\frac{1}{96}$			$\frac{5}{96}$				*$\frac{1}{32}$	*$\frac{1}{24}$			$\frac{5}{384}$	
7			$\frac{5}{768}$	$\frac{1}{120}$	*$\frac{1}{60}$		*$\frac{1}{256}$				$\frac{5}{192}$				*$\frac{1}{64}$				$\frac{5}{768}$	
9	*$\frac{1}{96}$		*$\frac{1}{256}$	$\frac{1}{32}$			*$\frac{5}{768}$		*$\frac{1}{24}$		*$\frac{1}{64}$				*$\frac{5}{192}$		*$\frac{1}{96}$		*$\frac{1}{256}$	*$\frac{1}{32}$
12				*$\frac{1}{32}$	*$\frac{1}{64}$	*$\frac{5}{192}$						$\frac{1}{32}$	$\frac{1}{64}$	$\frac{5}{192}$			*$\frac{3}{32}$	*$\frac{3}{64}$		
14				*$\frac{1}{24}$								$\frac{1}{24}$							$\frac{3}{64}$	
3, 3	*$\frac{1}{120}$	*$\frac{1}{60}$	*$\frac{1}{320}$				$\frac{3}{64}$													
5	$\frac{1}{160}$	*$\frac{1}{320}$						*$\frac{3}{64}$				*$\frac{5}{96}$	$\frac{5}{192}$	*$\frac{1}{64}$						$\frac{5}{96}$
8			*$\frac{5}{768}$	$\frac{1}{120}$	$\frac{1}{60}$		$\frac{1}{256}$				*$\frac{5}{192}$				$\frac{1}{64}$				*$\frac{5}{768}$	
10	*$\frac{1}{96}$		$\frac{1}{256}$	$\frac{1}{32}$			$\frac{5}{768}$		*$\frac{1}{24}$		$\frac{1}{64}$				$\frac{5}{192}$		*$\frac{1}{96}$		$\frac{1}{256}$	*$\frac{1}{32}$
11		*$\frac{1}{96}$	$\frac{1}{128}$		$\frac{1}{32}$		$\frac{5}{384}$			*$\frac{1}{24}$	$\frac{1}{32}$				$\frac{5}{96}$			*$\frac{1}{96}$	$\frac{1}{128}$	
13				*$\frac{1}{32}$	$\frac{1}{64}$	$\frac{5}{192}$						$\frac{1}{32}$	*$\frac{1}{64}$	*$\frac{5}{192}$			*$\frac{3}{32}$	$\frac{3}{64}$		
15				*$\frac{1}{24}$								$\frac{1}{24}$							*$\frac{3}{64}$	
16					*$\frac{1}{24}$								$\frac{1}{24}$						*$\frac{3}{32}$	
4, 1		$\frac{1}{160}$						*$\frac{3}{32}$					$\frac{5}{96}$	$\frac{1}{32}$						
2	$\frac{1}{160}$	$\frac{1}{320}$						$\frac{3}{64}$				$\frac{5}{96}$	$\frac{5}{192}$	*$\frac{1}{64}$						*$\frac{5}{96}$
4			$\frac{9}{320}$				$\frac{3}{64}$													
6		*$\frac{5}{384}$			*$\frac{9}{640}$	*$\frac{1}{384}$		$\frac{1}{128}$		$\frac{5}{96}$						$\frac{1}{32}$		*$\frac{5}{384}$		
7	*$\frac{5}{384}$	*$\frac{5}{768}$		*$\frac{9}{610}$	*$\frac{9}{1280}$	$\frac{1}{768}$		*$\frac{1}{256}$	*$\frac{5}{96}$	*$\frac{5}{192}$						*$\frac{1}{64}$	*$\frac{5}{384}$	*$\frac{5}{768}$		$\frac{5}{128}$
9	*$\frac{1}{128}$	$\frac{1}{256}$		*$\frac{1}{384}$	$\frac{1}{768}$	*$\frac{5}{256}$		*$\frac{5}{768}$	*$\frac{1}{32}$	$\frac{1}{64}$						*$\frac{5}{192}$	*$\frac{1}{128}$	$\frac{1}{256}$		*$\frac{1}{384}$
12																			$\frac{3}{64}$	
14				$\frac{1}{32}$	*$\frac{1}{64}$	*$\frac{5}{192}$						*$\frac{1}{32}$	$\frac{1}{64}$	$\frac{5}{192}$			*$\frac{3}{32}$	$\frac{3}{64}$		

4.2. $[321]\times[321]\rightarrow[321\alpha]_s+[321\beta]_a+[321\gamma]_a+[321\delta]_s+[321\varepsilon]_s$ (to be continued).

$[\nu]\beta$	$[321]\gamma$				$[321]\delta$								$[321]\varepsilon$							
m / m_1, m_2	7	9	12	14	1	2	4	6	7	9	12	14	1	2	4	6	7	9	12	14
1, 1					$\frac{1}{10}$								$\frac{1}{60}$							
2		$\frac{1}{24}$				*$\frac{1}{40}$	$\frac{3}{160}$				$\frac{1}{32}$			*$\frac{1}{240}$	$\frac{1}{320}$				$\frac{3}{64}$	
4	$\frac{5}{96}$	$\frac{1}{32}$				$\frac{3}{160}$						$\frac{1}{32}$		$\frac{1}{320}$						$\frac{3}{64}$
6								*$\frac{1}{10}$								*$\frac{1}{60}$				
7		$\frac{1}{32}$	*$\frac{1}{128}$	$\frac{1}{96}$					$\frac{1}{40}$						*$\frac{5}{192}$		$\frac{1}{240}$	*$\frac{1}{16}$	*$\frac{1}{64}$	$\frac{1}{48}$
9	$\frac{1}{32}$	$\frac{1}{120}$	$\frac{5}{384}$							$\frac{1}{40}$				*$\frac{1}{48}$	*$\frac{1}{64}$		*$\frac{1}{16}$	*$\frac{1}{240}$	$\frac{5}{192}$	
12				*$\frac{1}{160}$		$\frac{1}{32}$						*$\frac{3}{160}$		*$\frac{3}{64}$						$\frac{1}{320}$
14			*$\frac{1}{160}$	*$\frac{1}{120}$			$\frac{1}{32}$				*$\frac{3}{160}$	*$\frac{1}{40}$			*$\frac{3}{64}$				$\frac{1}{320}$	$\frac{1}{240}$
2, 1		*$\frac{1}{24}$				*$\frac{1}{40}$	$\frac{3}{160}$				$\frac{1}{32}$			*$\frac{1}{240}$	$\frac{1}{320}$				$\frac{3}{64}$	
2					*$\frac{1}{40}$	$\frac{1}{20}$	$\frac{3}{320}$				$\frac{1}{64}$		*$\frac{1}{240}$	$\frac{1}{120}$	$\frac{1}{640}$				$\frac{3}{128}$	
4	$\frac{5}{192}$	*$\frac{1}{64}$			$\frac{3}{160}$	$\frac{3}{320}$						*$\frac{1}{64}$	$\frac{1}{320}$	$\frac{1}{640}$						*$\frac{3}{128}$
6		*$\frac{1}{32}$	*$\frac{1}{128}$	*$\frac{1}{96}$					$\frac{1}{40}$						*$\frac{5}{192}$		$\frac{1}{240}$	$\frac{1}{16}$	*$\frac{1}{64}$	*$\frac{1}{48}$
7			*$\frac{1}{256}$					$\frac{1}{40}$	*$\frac{1}{20}$						*$\frac{5}{384}$	$\frac{1}{240}$	*$\frac{1}{120}$		*$\frac{1}{128}$	
9		$\frac{1}{60}$	*$\frac{5}{768}$							$\frac{1}{20}$			$\frac{1}{48}$		$\frac{1}{128}$	$\frac{1}{16}$		*$\frac{1}{120}$	*$\frac{5}{384}$	
12				$\frac{1}{320}$	$\frac{1}{32}$	$\frac{1}{64}$						$\frac{3}{320}$	*$\frac{3}{64}$	*$\frac{3}{128}$						*$\frac{1}{640}$
14			$\frac{1}{320}$	*$\frac{1}{60}$			*$\frac{1}{64}$				$\frac{3}{320}$	*$\frac{1}{20}$			$\frac{3}{128}$				*$\frac{1}{640}$	$\frac{1}{120}$
3, 3					*$\frac{1}{40}$	*$\frac{1}{20}$	*$\frac{3}{320}$				*$\frac{1}{64}$		*$\frac{1}{240}$	*$\frac{1}{120}$	*$\frac{1}{640}$				*$\frac{3}{128}$	
5	*$\frac{5}{192}$	$\frac{1}{64}$			$\frac{3}{160}$	*$\frac{3}{320}$						$\frac{1}{64}$	$\frac{1}{320}$	*$\frac{1}{640}$						$\frac{3}{128}$
8			$\frac{1}{256}$					$\frac{1}{40}$	$\frac{1}{20}$						$\frac{5}{384}$	$\frac{1}{240}$	$\frac{1}{120}$		$\frac{1}{128}$	
10		*$\frac{1}{60}$	$\frac{5}{768}$							*$\frac{1}{20}$			$\frac{1}{48}$		*$\frac{1}{128}$	$\frac{1}{16}$		$\frac{1}{120}$	$\frac{5}{384}$	
11	*$\frac{1}{32}$	$\frac{1}{120}$	$\frac{5}{384}$							$\frac{1}{40}$				$\frac{1}{48}$	*$\frac{1}{64}$		$\frac{1}{16}$	*$\frac{1}{240}$	$\frac{5}{192}$	
13				*$\frac{1}{320}$	$\frac{1}{32}$	*$\frac{1}{64}$						*$\frac{3}{320}$	*$\frac{3}{64}$	$\frac{3}{128}$						$\frac{1}{640}$
15			*$\frac{1}{320}$	$\frac{1}{60}$			$\frac{1}{64}$				*$\frac{3}{320}$	$\frac{1}{20}$			*$\frac{3}{128}$				$\frac{1}{640}$	*$\frac{1}{120}$
16			*$\frac{1}{160}$	*$\frac{1}{120}$			$\frac{1}{32}$				*$\frac{3}{160}$	*$\frac{1}{40}$			*$\frac{3}{64}$				$\frac{1}{320}$	$\frac{1}{240}$
4, 1	*$\frac{5}{96}$	*$\frac{1}{32}$				$\frac{3}{160}$						$\frac{1}{32}$		$\frac{1}{320}$						$\frac{3}{64}$
2	*$\frac{5}{192}$	$\frac{1}{64}$			$\frac{3}{160}$	$\frac{3}{320}$						*$\frac{1}{64}$	$\frac{1}{320}$	$\frac{1}{640}$						*$\frac{3}{128}$
4							$\frac{27}{320}$				*$\frac{1}{64}$				$\frac{9}{640}$				*$\frac{3}{128}$	
6	$\frac{5}{128}$	*$\frac{1}{384}$		$\frac{1}{128}$					$\frac{3}{160}$	*$\frac{1}{32}$				$\frac{5}{192}$			*$\frac{1}{20}$			$\frac{1}{64}$
7	$\frac{5}{256}$	$\frac{1}{768}$		*$\frac{1}{256}$				$\frac{3}{160}$	$\frac{3}{320}$	$\frac{1}{64}$			$\frac{5}{192}$	$\frac{5}{384}$		*$\frac{1}{20}$	*$\frac{1}{40}$			*$\frac{1}{128}$
9	$\frac{1}{768}$	$\frac{9}{1280}$		*$\frac{5}{768}$				*$\frac{1}{32}$	$\frac{1}{64}$	*$\frac{3}{320}$			$\frac{1}{64}$	*$\frac{1}{128}$				*$\frac{1}{40}$		*$\frac{5}{384}$
12			*$\frac{9}{320}$				*$\frac{1}{64}$				*$\frac{27}{320}$				$\frac{3}{128}$				$\frac{9}{640}$	
14				*$\frac{1}{320}$	$\frac{1}{32}$	*$\frac{1}{64}$						*$\frac{3}{320}$	*$\frac{3}{64}$	$\frac{3}{128}$						$\frac{1}{640}$

4.3. $[321]\times[321]\rightarrow[321\alpha]_s+[321\beta]_a+[321\gamma]_a+[321\delta]_s+[321\varepsilon]_s$ (to be continued).

$[\nu]\beta$	[321]α								[321]β								[321]γ			
m_1, m_2 \ m	1	2	4	6	7	9	12	14	1	2	4	6	7	9	12	14	1	2	4	6
5, 3	1/160	*1/320						*3/64				5/96	*5/192	1/64						*5/96
5			*9/320				*3/64													
8	*5/384	5/768		*9/640	9/1280	*1/768		1/256	*5/96	5/192						1/64	*5/384	5/768		5/128
10	*1/128	*1/256		*1/384	*1/768	5/256		5/768	*1/32	*1/64						5/192	*1/128	*1/256		*1/384
11		1/128			1/384	5/128		5/384		1/32						5/96		1/128		
13																			*3/64	
15				1/32	1/64	5/192						*1/32	*1/64	*5/192			*3/32	*3/64		
16					*1/32	5/96							1/32	*5/96				3/32		
6, 1				*1/30																
2			5/384		1/120	1/32	*1/128	*1/96			*5/96				1/32	1/24			*5/384	
4		*5/384			*9/640	*1/384		1/128		5/96						*1/32		5/384		
6	*1/30																			
7		1/120	*1/10											1/24						
9		1/8											*1/24							
12		*1/128			*1/384	*5/128		*5/384		*1/32						*5/96		1/128		
14		*1/96	1/128		1/32		5/384			*1/24	1/32				5/96			1/96	*1/128	
7, 1			5/384		1/120	*1/32	*1/128	1/96			*5/96				1/32	*1/24			*5/384	
2			5/768	1/120	*1/60		*1/256				*5/192				1/64				*5/768	
4	*5/384	*5/768		*9/640	*9/1280	1/768		*1/256	5/96	5/192						1/64	5/384	5/768		*5/128
6		1/120	*1/10											*1/24						
7	1/120	*1/60	*1/20																	
9	*1/8											1/24								
12	*1/128	*1/256		*1/384	*1/768	5/256		5/768	*1/32	*1/64						5/192	1/128	1/256		1/384
14	1/96		*1/256	*1/32			*5/768		1/24		*1/64				*5/192		*1/96		1/256	*1/32
8, 3			*5/768	1/120	1/60		1/256				5/192				*1/64				5/768	
5	*5/384	5/768		*9/640	9/1280	*1/768		1/256	5/96	*5/192						*1/64	5/384	*5/768		*5/128
8	1/120	1/60	1/20																	
10	*1/8											1/24								
11		*1/8											1/24							
13	*1/128	1/256		*1/384	1/768	*5/256		*5/768	*1/32	1/64						*5/192	1/128	*1/256		1/384
15	1/96		1/258	*1/32			5/768		1/24		1/64				5/192		*1/96		*1/256	*1/32
16		1/96	1/128		*1/32		5/384			1/24	1/32				5/96			*1/96	*1/128	

4.4. $[321]\times[321]\rightarrow[321\alpha]_s+[321\beta]_a+[321\gamma]_a+[321\delta]_s+[321\varepsilon]_s$.

$[\nu]\beta$	$[321]\gamma$				$[321]\delta$								$[321]\varepsilon$							
m_1, m_2 \ m	7	9	12	14	1	2	4	6	7	9	12	14	1	2	4	6	7	9	12	14
5,3	$\frac{5}{192}$	$*\frac{1}{64}$			$\frac{3}{160}$	$*\frac{3}{320}$						$\frac{1}{64}$	$\frac{1}{320}$	$*\frac{1}{540}$						$\frac{3}{128}$
5							$*\frac{27}{320}$				$\frac{1}{64}$				$*\frac{9}{640}$				$\frac{3}{128}$	
8	$*\frac{5}{256}$	$*\frac{1}{768}$		$\frac{1}{256}$				$\frac{3}{160}$	$*\frac{3}{320}$	$*\frac{1}{64}$			$\frac{5}{192}$	$*\frac{5}{384}$		$*\frac{1}{20}$	$\frac{1}{40}$			$\frac{1}{128}$
10	$*\frac{1}{768}$	$*\frac{9}{1280}$		$\frac{5}{768}$				$*\frac{1}{32}$	$*\frac{1}{64}$	$\frac{3}{320}$			$\frac{1}{64}$	$\frac{1}{128}$				$\frac{1}{40}$		$\frac{5}{384}$
11	$\frac{1}{384}$	$*\frac{9}{640}$		$\frac{5}{384}$					$\frac{1}{32}$	$\frac{3}{160}$				$*\frac{1}{64}$				$\frac{1}{20}$		$\frac{5}{192}$
13			$\frac{9}{320}$				$\frac{1}{64}$				$\frac{27}{320}$				$*\frac{3}{128}$				$*\frac{9}{640}$	
15				$\frac{1}{320}$	$\frac{1}{32}$	$\frac{1}{64}$						$\frac{3}{320}$	$*\frac{3}{64}$	$*\frac{3}{128}$						$*\frac{1}{640}$
16				$\frac{1}{160}$		$*\frac{1}{32}$						$\frac{3}{160}$		$\frac{3}{64}$						$*\frac{1}{320}$
6, 1								$*\frac{1}{10}$								$*\frac{1}{60}$				
2		$\frac{1}{32}$	$\frac{1}{128}$	$\frac{1}{96}$					$\frac{1}{40}$						$*\frac{5}{192}$		$\frac{1}{240}$	$\frac{1}{16}$	$*\frac{1}{64}$	$*\frac{1}{48}$
4	$*\frac{5}{128}$	$\frac{1}{384}$		$*\frac{1}{128}$					$\frac{3}{160}$	$*\frac{1}{32}$				$\frac{5}{192}$			$*\frac{1}{20}$			$\frac{1}{64}$
6					$*\frac{1}{10}$								$*\frac{1}{60}$							
7				$\frac{1}{8}$		$\frac{1}{40}$	$\frac{3}{160}$				$\frac{1}{32}$			$\frac{1}{240}$	$\frac{1}{320}$				$*\frac{1}{192}$	
9			$*\frac{1}{10}$	$*\frac{1}{120}$			$*\frac{1}{32}$				$\frac{3}{160}$	$*\frac{1}{40}$			$*\frac{1}{192}$				$*\frac{1}{320}$	$\frac{1}{240}$
12	$\frac{1}{384}$	$*\frac{9}{640}$		$\frac{5}{384}$					$\frac{1}{32}$	$\frac{3}{160}$				$*\frac{1}{64}$				$\frac{1}{20}$		$\frac{5}{192}$
14	$\frac{1}{32}$	$*\frac{1}{120}$	$*\frac{5}{384}$							$*\frac{1}{40}$				$*\frac{1}{48}$	$\frac{1}{64}$		$*\frac{1}{16}$	$\frac{1}{240}$	$*\frac{5}{192}$	
7, 1		$*\frac{1}{32}$	$\frac{1}{128}$	$*\frac{1}{96}$					$\frac{1}{40}$						$*\frac{5}{192}$		$\frac{1}{240}$	$*\frac{1}{16}$	$*\frac{1}{64}$	$\frac{1}{48}$
2			$\frac{1}{256}$					$\frac{1}{40}$	$*\frac{1}{20}$						$*\frac{5}{384}$	$\frac{1}{240}$	$*\frac{1}{120}$		$*\frac{1}{128}$	
4	$*\frac{5}{256}$	$*\frac{1}{768}$		$\frac{1}{256}$				$\frac{3}{160}$	$\frac{3}{320}$	$\frac{1}{64}$			$\frac{5}{192}$	$\frac{5}{384}$		$*\frac{1}{20}$	$*\frac{1}{40}$			$*\frac{1}{128}$
6				$*\frac{1}{8}$		$\frac{1}{40}$	$\frac{3}{160}$				$\frac{1}{32}$			$\frac{1}{240}$	$\frac{1}{320}$				$*\frac{1}{192}$	
7					$\frac{1}{40}$	$*\frac{1}{20}$	$\frac{3}{320}$				$\frac{1}{64}$		$\frac{1}{240}$	$*\frac{1}{120}$	$\frac{1}{640}$				$*\frac{1}{384}$	
9			$\frac{1}{20}$	$*\frac{1}{60}$			$\frac{1}{64}$				$*\frac{3}{320}$	$*\frac{1}{20}$			$\frac{1}{384}$				$\frac{1}{640}$	$\frac{1}{120}$
12	$\frac{1}{768}$	$\frac{9}{1280}$		$*\frac{5}{768}$				$\frac{1}{32}$	$\frac{1}{64}$	$*\frac{3}{320}$			$*\frac{1}{64}$	$*\frac{1}{128}$				$*\frac{1}{40}$		$*\frac{5}{384}$
14		$*\frac{1}{60}$	$\frac{5}{768}$							$*\frac{1}{20}$			$\frac{1}{48}$		$*\frac{1}{128}$	$\frac{1}{16}$		$\frac{1}{120}$	$\frac{5}{384}$	
8, 3			$*\frac{1}{256}$					$\frac{1}{40}$	$\frac{1}{20}$						$\frac{5}{384}$	$\frac{1}{240}$	$\frac{1}{120}$		$\frac{1}{128}$	
5	$\frac{5}{256}$	$\frac{1}{768}$		$*\frac{1}{256}$				$\frac{3}{160}$	$*\frac{3}{320}$	$*\frac{1}{64}$			$\frac{5}{192}$	$*\frac{5}{384}$		$*\frac{1}{20}$	$\frac{1}{40}$			$\frac{1}{128}$
8					$\frac{1}{40}$	$\frac{1}{20}$	$*\frac{3}{320}$				$*\frac{1}{64}$		$\frac{1}{240}$	$\frac{1}{120}$	$*\frac{1}{640}$				$\frac{1}{384}$	
10			$*\frac{1}{20}$	$\frac{1}{60}$			$*\frac{1}{64}$				$\frac{3}{320}$	$\frac{1}{20}$			$*\frac{1}{384}$				$*\frac{1}{640}$	$*\frac{1}{120}$
11			$*\frac{1}{10}$	$*\frac{1}{120}$			$*\frac{1}{32}$				$\frac{3}{160}$	$*\frac{1}{40}$			$*\frac{1}{192}$				$*\frac{1}{320}$	$\frac{1}{240}$
13	$*\frac{1}{768}$	$*\frac{9}{1280}$		$\frac{5}{768}$				$\frac{1}{32}$	$*\frac{1}{64}$	$\frac{3}{320}$			$*\frac{1}{64}$	$\frac{1}{128}$				$\frac{1}{40}$		$\frac{5}{384}$
15		$\frac{1}{60}$	$*\frac{5}{768}$							$\frac{1}{20}$			$\frac{1}{48}$		$\frac{1}{128}$	$\frac{1}{16}$		$*\frac{1}{120}$	$*\frac{5}{384}$	
16	$*\frac{1}{32}$	$*\frac{1}{120}$	$*\frac{5}{384}$							$*\frac{1}{40}$				$\frac{1}{48}$	$\frac{1}{64}$		$\frac{1}{16}$	$\frac{1}{240}$	$*\frac{5}{192}$	

Table 5. The sign factor for the Yamanouchi basis of S(f)†.

S(3)	[21]	1 2	+ −
S(4)	[31]	1 2 3	+ − +
S(4)	[22]	1 2	+ −
S(4)	[211]	1 2 3	+ − +
S(5)	[41]	1 2 3 4	+ − + −
S(5)	[32]	1 2 3 4 5	+ − + + −
S(5)	[311]	1 2 3 4 5 6	+ − + + − −
S(5)	[221]	1 2 3 4 5	+ − − + −
S(5)	$[21^3]$	1 2 3 4	+ − + −
S(6)	[51]	1 2 3 4 5	+ − + − +
S(6)	[42]	1 2 3 4 5 6 7 8 9	+ − + − + − + + −
S(6)	[411]	1 2 3 4 5 6 7 8 9 10	+ − + − + − + + − +
S(6)	[33]	1 2 3 4 5	+ − + + −
S(6)	[321]	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16	+ − + + − − + − − + − − + + − +
S(6)	$[31^3]$	1 2 3 4 5 6 7 8 9 10	+ − + + − + − + − +
S(6)	$[2^3]$	1 2 3 4 5	+ − − + −
S(6)	$[2^2 1^2]$	1 2 3 4 5 6 7 8 9	+ − − + − + − + −
S(6)	$[21^4]$	1 2 3 4 5	+ − + − +

† The decreasing page order for the Yamanouchi symbols $(r_f r_{f-1} \ldots r_2 r_1)$ is used.

5. Discussion

The present paper together with Chen *et al* (1984b) gives a complete listing of the CG coefficients for the permutation groups S(2)–S(6), referred to as the new CG coefficient tables.

Our CG coefficients differ from the Schindler and Mirman result in the following respects: (a) Ours have a consistent absolute phase convention while theirs do not. (b) Our multiplicity separation is based, whenever possible, on the imposition of the symmetries, while theirs is based on an *ad hoc* choice. (c) Our tables give the exact values of the coefficients instead of the approximate values with 16 decimal places.

The new CG coefficient tables differ from the old ones (Chen and Gao 1981) merely in the absolute phase, except in the supplement of table 4. To reconcile the two results, the absolute phases of the CG coefficients should be reversed for the following $[\sigma]\times[\mu]\rightarrow[\nu]$ in the old table 1.1-4.14 of Chen and Gao (1981).

Table	
1.1	[21]×[21]→[3]+[21]
2.1	[31]×[31]→[31]
2.3	[22]×[22]→[22]
3.1	[41]×[41]→[41]+[32]+[311]
3.4	[32]×[32]→[41]+[32]+$[21^3]$
3.5	[311]×[32]→[221]+$[21^3]$
3.6	[311]×[311]→[41]+[311]+[221β]
4.1	[51]×[51]→[51]+[42]+[411]
4.2	[51]×[42]→[42]+[411]+[33]+[321]
4.4	[51]×[33]→[321]
4.5	[33]×[33]→[42]
4.6	[42]×[33]→[33]
4.7	[411]×[33]→$[2^3]$
4.8	[42]×[42]→[51]+[411]+[321α]+$[2^3]$
4.9	[42]×[411]→$[31^3]$
4.10	[411]×[411]→[411]+[321α]+[321β]+$[2^3]$
4.11	[51]×[321]→[411]+[321β]+$[2^3]$
4.12	[33]×[321]→[411]+[321α]+[321β]
4.13	[42]×[321]→[321α]
4.14	[411]×[321]→[321α]+[321γ]

The CG coefficients for [321]×[321]→[321] have a multiplicity five, the highest for the permutation group S(6). Without the symmetry imposition (3.11) or (4.8), the computer produced CG coefficients are totally unsuitable for being put in the square root form. After the symmetry imposition (4.8), the first three sets ($\tau=\alpha,\beta,\gamma$) of CG coefficients are fairly simple but the last two sets ($\tau=\delta,\varepsilon$) are still not simple enough with something like $(243/3200)^{1/2}$ appearing. Then we noticed from (4.8) that these two sets have identical symmetries under the transposition π_{12} and the tilde operation $\mathscr{C}_{12}$, and any linear combination of them will not affect these symmetries. This freedom can be exploited to simplify the coefficients. Notice that although after the imposition of the symmetries (3.11) we cannot have the symmetry

$$C^{[321]\tau,m_3}_{[321]m_1,[321]m_2}=\Delta_\tau\Lambda^{[321]}_{m_1}\Lambda^{[321]}_{m}C^{[321]\tau,\tilde{m}_1}_{[321]\tilde{m}_3,[321]m_2}, \qquad (5.1)$$

due to (3.3*d*), maybe we can still have the 'broken' symmetry

$$C^{[321]\tau,m_3}_{[321]m_1,[321]m_2}=\Delta(\tau,m_1,m_2,m)C^{[321]\tau,\tilde{m}_1}_{[321]\tilde{m}_3,[321]m_2}, \qquad \tau=\delta,\varepsilon, \qquad (5.2)$$

where $\Delta(\tau,m_1,m_2,m)=\pm1$ does not only depend on τ but also on m_1, m_2 and m. We tried to impose the 'broken' symmetry (5.2) for the δ- and εth sets of CG coefficients and really got much simpler result as shown in table 4. From (4.8) it is seen that for the first three sets ($\tau=\alpha,\beta,\gamma$) no such freedom exists and therefore we have got as many symmetries as possible.

From the above example we see that the nature seems to be in favour of the symmetry. The more symmetries (including the 'broken' one) are imposed, the more simple coefficients resulted. One may naturally inquire what is the simplest result and whether there is anything, such as some yet unknown quantum numbers, behind this simplicity? These are the long standing open questions for the multiplicity separation of the CG coefficient.

The eigenfunction method is a 'packaged method', i.e. it is impossible to get a particular CG coefficient without producing the whole CG matrix. Therefore the limitation of the method is decided by the size of the CG matrix versus the size of the RAM (randomly accessible memory) of the computer. Another limitation is that for higher multiplicity, the symmetry imposition becomes more and more involved. Without proper symmetry imposition, we cannot convert the coefficient into a simple square root form. However, if we are content with the decimal form of the coefficients, then we are free from this limitation.

The $U(mn) \supset U(m) \times U(n)$ f_2-particle CFP for an f $(=f_1+f_2)$-particle system can be calculated from the transformation coefficients of $S(f)$, which are tabulated by Chen *et al* (1983b), and the CG coefficients of $S(f_1)$, $S(f_2)$ and $S(f)$ (see equation (17) and also examples in Chen *et al* (1983c)). It was shown in Chen *et al* 1984b that under the phase convention of § 3, the $U(mn) \supset U(m) \times U(n)$ CFP calculated from the permutation group CG coefficients is also the $SU(mn) \supset SU(m) \times SU(n)$ CFP. The $SU(mn) \supset SU(m) \times SU(n)$ one-particle CFP have been tabulated by Chen *et al* (1984b), while the two- and three-particle CFP will be published in the forthcoming papers.

Let $|Y^{\nu_1}_{m_1} W_1\rangle$ ($|Y^{\nu_2}_{m_2} W_2\rangle$) be the Yamanouchi basis $\nu_1 m_1(\nu_2 m_2)$ of the permutation group $S(f)$, as well as the IRB $\nu_1 W_1(\nu_2 W_2)$ of the unitary group $SU(m)$ ($SU(n)$) in the $x(\xi)$ space, where $Y^{\nu_i}_{m_i}$ stands for the Young tableau and W_i the component index for the irrep ν_i of the unitary group $SU(m)$ or $SU(n)$. The $SU(mn) \supset SU(m) \times SU(n)$ IRB and the Yamanouchi basis of $S(f)$ in the (x, ξ) space can be constructed in terms of the permutation group CG coefficients (Chen *et al* 1978)

$$|Y^{\nu}_{m,\beta\nu_1} {}^{[\nu]}_{W_1\nu_2 W_2}\rangle = \eta_\nu \sum_{m_1 m_2} C^{[\nu]\beta,m}_{\nu_1 m_1,\nu_2 m_2} |Y^{\nu_1}_{m_1} W_1\rangle |Y^{\nu_2}_{m_2} W_2\rangle,$$

where η_ν is a phase factor and can be chosen to be unit. Under this choice, the permutation group CG coefficient is identified to the indirect coupling coefficients for the $SU(mn) \supset SU(m) \times SU(n)$ IRB, and the $S(f) \supset S(f_1) \times S(f_2)$ ISF to the $SU(mn) \supset SU(m) \times SU(n)$ f_2-particle CFP (Chen 1981).

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